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2000 J. Phys. A: Math. Gen. 33 227

(http://iopscience.iop.org/0305-4470/33/1/313)

J. Phys. A: Math. Gen. **33** (2000) 227–232. Printed in the UK PII: S0305-4470(00)07162-6

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A R Usha Devi

Department of Physics, University of Bangalore, Jnanabharathi Campus, Bangalore—560056, India

Received 26 August 1999

Abstract. A hidden overall factor of $2\log_2(2s+1)$ is detected, when the co-planar Braunstein–Caves (BC) inequality for Einstein–Podolsky–Rosen–Bohm spin-s entanglement is expressed in terms of an information theoretic index of correlation. It is observed that the size of violation of the *normalized* co-planar BC inequality—which is defined by eliminating the overall factor $2\log_2(2s+1)$ from the original BC inequality—decreases with the increase of the spin value s, thus exhibiting a satisfactory behaviour in the classical limit.

Entropic Bell inequalities (Braunstein and Caves 1988, 1990, Cerf and Adami 1997, Wodkiewicz 1995) have been used to highlight the non-classical nature of quantum entanglement. The usefulness of these information theoretic inequalities is that they are applicable to any pair of entangled systems—not just two-state systems, as formulated in the case of usual correlation Bell inequalities, such as e.g., Clauser–Horne–Shimony–Holt inequalities (Clauser *et al* 1969). It was Braunstien and Caves (1988) who first formulated the information theoretic Bell inequalities, which apply to any pair of spatially separated entangled physical systems. They observed that the Einstein–Podolsky–Rosen–Bohm (EPRB) (Bohm 1951, Einstein *et al* 1935) spin-s correlations violate these information theoretic inequalities for all values of spin s, for a specified co-planar geometry of analyser orientations. It has been observed (Braunstein and Caves 1988, 1990) that the strength of violation grows as spin s increases, even though the range of analyser orientation angles, over which the violation is observed, decreases with the increase of s. According to Braunstein and Caves, '*The biggest surprise...is not the presence of violation for all s, but rather the increasing size of violation as s increases...*' (Braunstein and Caves 1990).

In this paper we identify that there is a hidden overall factor $2\log_2(2s+1)$ in the co-planar BC inequalities, which leads to the increasing size of violation with the increase of s. We show that the normalized co-planar BC inequalities, obtained by eliminating this overall factor, give rise to the opposite result, namely, decreasing size of violation as spin s increases.

We construct joint information entropy (in bits) $H(\vec{a},\vec{b})$ for EPRB spin-s correlations through

$$H(\vec{a},\vec{b}) = -\sum_{\lambda_a,\lambda_b=-s}^{s} \mathcal{P}^s(\lambda_a,\lambda_b;\theta)\log_2 \mathcal{P}^s(\lambda_a,\lambda_b;\theta) \tag{1}$$

where

$$\mathcal{P}^s(\lambda_a,\lambda_b;\theta) = \frac{1}{(2s+1)^2}\sum_{k=0}^{2s}(-1)^k(2k+1)c(sks;\lambda_a0\lambda_a)c(sks;\lambda_b0\lambda_b)P_k(\cos\theta) \tag{2}$$

are the quantum mechanical non-local (analyser-dependent) joint probabilities (Mermin 1980, Usha Devi *et al* 1997) that the spin component $\hat{\hat{\mathrm{S}}}_1 \cdot \vec{\mathrm{a}}$ of particle 1 is λ_a along the analyser orientation $\vec{\mathrm{a}}$ and the spin component $\hat{\hat{\mathrm{S}}}_2 \cdot \vec{\mathrm{b}}$ of particle 2 is λ_b along the analyser orientation $\vec{\mathrm{b}}$; $c(sks; \lambda 0 \lambda)$ denote the Clebsch–Gordan coefficients and θ is the angle between the analyser orientations $\vec{\mathrm{a}}$ and $\vec{\mathrm{b}}$. The individual probabilities $\mathcal{P}^s(\lambda_a), \mathcal{P}^s(\lambda_b)$, which govern the isolated measurements made independently on particles 1 and 2 can be derived (Usha Devi *et al* 1997) from the joint probabilities:

$$\begin{aligned} \mathcal{P}^s(\lambda_a) &= \sum_{\lambda_b=-s}^{s} \mathcal{P}^s(\lambda_a, \lambda_b; \theta) = \frac{1}{(2s+1)} \\ \mathcal{P}^s(\lambda_b) &= \sum_{\lambda_a=-s}^{s} \mathcal{P}^s(\lambda_a, \lambda_b; \theta) = \frac{1}{(2s+1)}. \end{aligned} \tag{3}$$

In statistical terminology (Feller 1967), $\mathcal{P}^s(\lambda_a), \mathcal{P}^s(\lambda_b)$ are referred to as marginal probabilities, since they are realized as marginals of the joint probabilities $\mathcal{P}^s(\lambda_a, \lambda_b; \theta)$.

The joint information entropy $H(\vec{\mathrm{a}}, \vec{\mathrm{b}})$ gives the total information carried jointly by the spin components $\hat{\hat{\mathrm{S}}}_1 \cdot \vec{\mathrm{a}}$ and $\hat{\hat{\mathrm{S}}}_2 \cdot \vec{\mathrm{b}}$. The information $H(\vec{\mathrm{a}})$ and $H(\vec{\mathrm{b}})$ carried separately by $\hat{\hat{\mathrm{S}}}_1 \cdot \vec{\mathrm{a}}$ and $\hat{\hat{\mathrm{S}}}_2 \cdot \vec{\mathrm{b}}$ are defined using the marginal probabilities $\mathcal{P}^s(\lambda_a), \mathcal{P}^s(\lambda_b)$:

$$\begin{aligned} H(\vec{\mathrm{a}}) &= -\sum_{\lambda_a=-s}^{s} \mathcal{P}^s(\lambda_a) \log_2 \mathcal{P}^s(\lambda_a) = \log_2(2s+1), \\ H(\vec{\mathrm{b}}) &= -\sum_{\lambda_b=-s}^{s} \mathcal{P}^s(\lambda_b) \log_2 \mathcal{P}^j(\lambda_b) = \log_2(2s+1). \end{aligned} \tag{4}$$

The conditional information entropy $H(\vec{\mathrm{a}}|\vec{\mathrm{b}})$ gives the information carried by the spin component $\hat{\hat{\mathrm{S}}}_1 \cdot \vec{\mathrm{a}}$ under the condition that $\hat{\hat{\mathrm{S}}}_2 \cdot \vec{\mathrm{b}}$ has assumed a certain value, and is defined by

$$\begin{aligned} H(\vec{\mathrm{a}}|\vec{\mathrm{b}}) &= -\sum_{\lambda_a,\lambda_b=-s}^{s} \mathcal{P}^s(\lambda_a, \lambda_b; \theta) \log_2 \mathcal{P}^s(\lambda_a|\lambda_b; \theta) \\ &= H(\vec{\mathrm{a}}, \vec{\mathrm{b}}) - H(\vec{\mathrm{b}}) \\ &= H(\vec{\mathrm{a}}, \vec{\mathrm{b}}) - \log_2(2s+1). \end{aligned} \tag{5}$$

In the above equation we have used the Bayes theorem (Feller 1968)

$$\mathcal{P}^s(\lambda_a|\lambda_b; \theta) = \frac{\mathcal{P}^s(\lambda_a, \lambda_b; \theta)}{\mathcal{P}^s(\lambda_b)} = (2s+1)\mathcal{P}^s(\lambda_a, \lambda_b; \theta) \tag{6}$$

for the conditional probabilities $\mathcal{P}^s(\lambda_a|\lambda_b; \theta)$.

It has been realized (Barnett and Phoenix 1989) that the mutual information entropy, i.e., the average information carried in common by the subsystems A and B of an entangled system, given by

$$H(A; B) = H(A) + H(B) - H(A, B) \tag{7}$$

where $H(A)$, $H(B)$ and $H(A, B)$ denote, respectively, the subsystem entropies and the joint entropy, serves as an information theoretic index of correlation. The mutual information entropy is zero when the joint probability $P(A, B) = P(A)P(B) \Longrightarrow H(A, B) = H(A)+H(B)$, i.e. only when the subsystems are statistically independent. If the subsystems are labelled such that $H(B) \geqslant H(A)$, the triangular inequalities (Wehrl 1978) for the information entropies,

$$|H(A) - H(B)| \leqslant H(A, B) \leqslant H(A) + H(B) \tag{8}$$

lead to the following bounds for the mutual information entropy:

$$0 \leqslant H(A;B) \leqslant 2H(A). \tag{9}$$

Thus, a normalized correlation index $0 \leqslant H_N(A;B) \leqslant 1$ can be defined as

$$H_N(A;B) = \frac{H(A) + H(B) - H(A,B)}{2H(A)}. \tag{10}$$

For EPRB spin correlations, the normalized index of correlation $H_N(\vec{\mathrm{a}};\vec{\mathrm{b}}) \equiv I^s(\theta)$ is given by

$$\begin{aligned} I^s(\theta) &= \frac{2\log_2(2s+1) - H(\vec{\mathrm{a}},\vec{\mathrm{b}})}{2\log_2(2s+1)} \\ &= \frac{1}{2} - \frac{H(\theta)}{2\log_2(2s+1)} \end{aligned} \tag{11}$$

where we have used equation (5) to express joint information entropy $H(\vec{\mathrm{a}},\vec{\mathrm{b}})$ in terms of conditional entropy $H(\vec{\mathrm{a}}|\vec{\mathrm{b}}) \equiv H(\theta)$. We have plotted $I^s(\theta)$ in figure 1 as a function of the angle θ between the analyser orientations, for spin values $s = \frac{1}{2}, 1, \frac{3}{2}$ and 2. We observe that the correlation index reaches the maximum value $\frac{1}{2}$ for parallel and anti-parallel analyser orientations. The increasing trend of $I^s(\theta)$ for $\theta \to 0^\circ$ and $\theta \to 180^\circ$ highlights the strong correlation between the spins for nearly parallel and nearly anti-parallel analyser orientations. In the large-s, small-θ limit, the index of correlation $I^s(\theta)$ assumes the form

$$I^s(\theta) \sim \frac{1}{2} - \frac{(s\theta)^2}{6\log_2(2s+1)}\left(\log_2\frac{1}{(s\theta)^2} + \frac{8}{3}\log_2 e\right) \tag{12}$$

where it can be clearly seen that $I^s(\theta) \to \frac{1}{2}$ as $\theta \to 0^\circ$.

We now take up the information theoretic Braunstein–Caves (BC) inequalities which involve the conditional information entropies in the form

$$H(\vec{\mathrm{a}}|\vec{\mathrm{b}}) \leqslant H(\vec{\mathrm{a}}|\vec{\mathrm{b}}') + H(\vec{\mathrm{b}}'|\vec{\mathrm{a}}') + H(\vec{\mathrm{a}}'|\vec{\mathrm{b}}) \tag{13}$$

where $\vec{\mathrm{a}}, \vec{\mathrm{b}}', \vec{\mathrm{a}}', \vec{\mathrm{b}}$ denote orientations of the analysers corresponding to particles 1 and 2 in four different sets of experimental runs. The BC inequalities dictate that the subsystem of any entangled system must carry an amount of information to be consistent with the local realistic theory. However, quantum correlations involving a pair of spin-s particles in a singlet state are observed (Braunstein and Caves 1988, 1990) to violate BC inequalities and hence are inconsistent with local realism. For the special case $\vec{\mathrm{a}}, \vec{\mathrm{b}}', \vec{\mathrm{a}}'$ and $\vec{\mathrm{b}}$ are co-planar and when the successive vectors in the list are separated by an angle $\frac{\theta}{3}$ (i.e., $\vec{\mathrm{a}}\cdot\vec{\mathrm{b}}' = \vec{\mathrm{a}}'\cdot\vec{\mathrm{b}} = \vec{\mathrm{a}}'\cdot\vec{\mathrm{b}} = \cos\frac{\theta}{3}$ and $\vec{\mathrm{a}}\cdot\vec{\mathrm{b}} = \cos\theta$), the BC inequality is violated if

$$\mathcal{H}(\theta) \equiv H(\theta) - 3H\left(\frac{\theta}{3}\right) \tag{14}$$

is positive.

It has been observed that the co-planar BC inequality is violated for all values of s. The violation is often attributed to the tight correlation between spins in the region of violation. With a view to examine the violation in terms of the strength of correlation, we express the conditional entropy $H(\theta) = \log_2(2s+1)(1 - 2I^s(\theta))$, in terms of the index of correlation (using equation (11)) in the co-planar BC inequality to obtain

$$2\log_2(2s+1)\left[3I^s\left(\frac{\theta}{3}\right) - I^s(\theta)\right] - 2\log_2(2s+1) \leqslant 0. \tag{15}$$

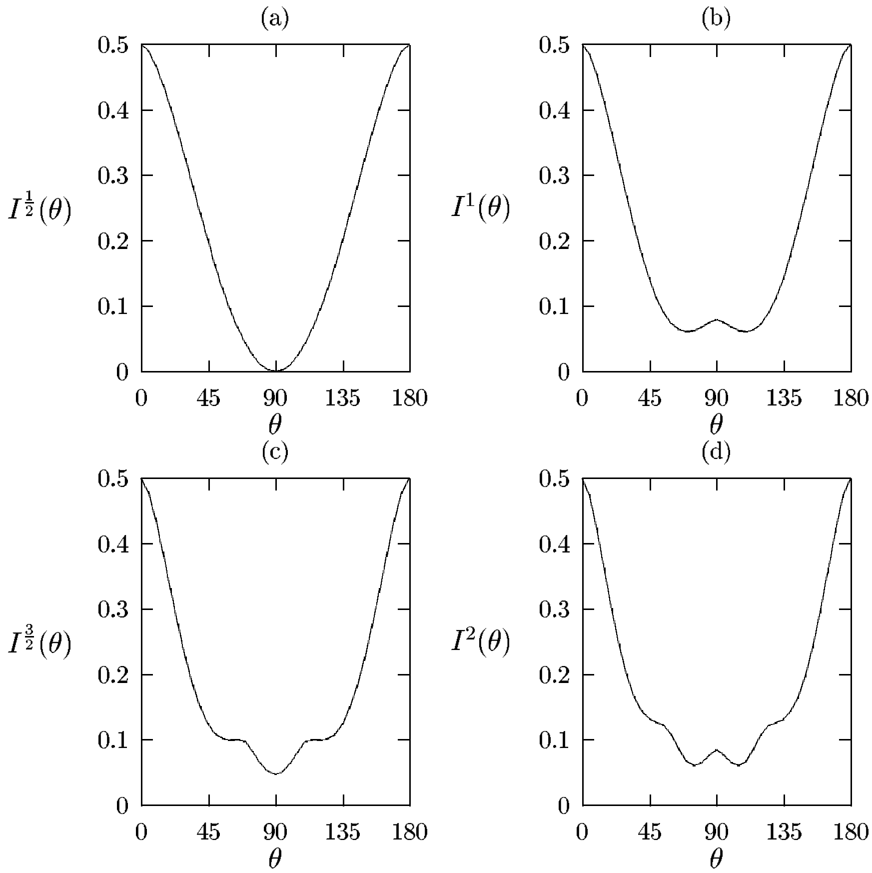

Figure 1. Information theoretic index of correlation $I^s(\theta)$ for EPRB spin-s correlations as a function of the analyser orientation angle θ. (*a*) spin-$\frac{1}{2}$, (*b*) spin-1, (*c*) spin-$\frac{3}{2}$, (*d*) spin-2.

Note that in the above equation, there is an overall factor $2\log_2(2s+1)$ which increases with the increase of spin value s. We define the *'normalized'* BC inequality for co-planar geometry through

$$\mathcal{I}^s(\theta) \equiv 3I^s\left(\frac{\theta}{3}\right) - I^s(\theta) - 1 \leqslant 0. \tag{16}$$

As the range of violation of any inequality remains unaltered by multiplying it throughout by a positive number, the *normalized* BC inequality is expected to be violated in the range of angles θ for which $\mathcal{H}(\theta)$ (of equation (14)) is positive. However, from the form of equation (16) it could be observed that $\mathcal{I}^s(\theta)$ cannot be positive if $I^s(\frac{\theta}{3}) < \frac{1}{3}$. This observation leads to the identification of a *tight correlation domain* as one in which the index of correlation $I^s > \frac{1}{3}$. Note that a variation of the index of correlation from $\frac{1}{3}+\Delta$ to δ, (Δ, δ are positive and are bound through $0 \leqslant I^s \leqslant \frac{1}{2}$) when the analyser orientations change from $\frac{\theta}{3}$ to θ, leads to the violation of equation (16), if $3\Delta - \delta > 0$.

In order to verify the variation of the size of violation with the increase of s we have plotted $\mathcal{I}^s(\theta)$ in figure 2 as a function of the angle θ for spin values $s = \frac{1}{2}, 1, \frac{3}{2}$ and 2, in the region of violation. The important feature that could be noted from figure 2 is that *the strength of violation decreases with the increase of spin value s*. We, therefore, realize that it

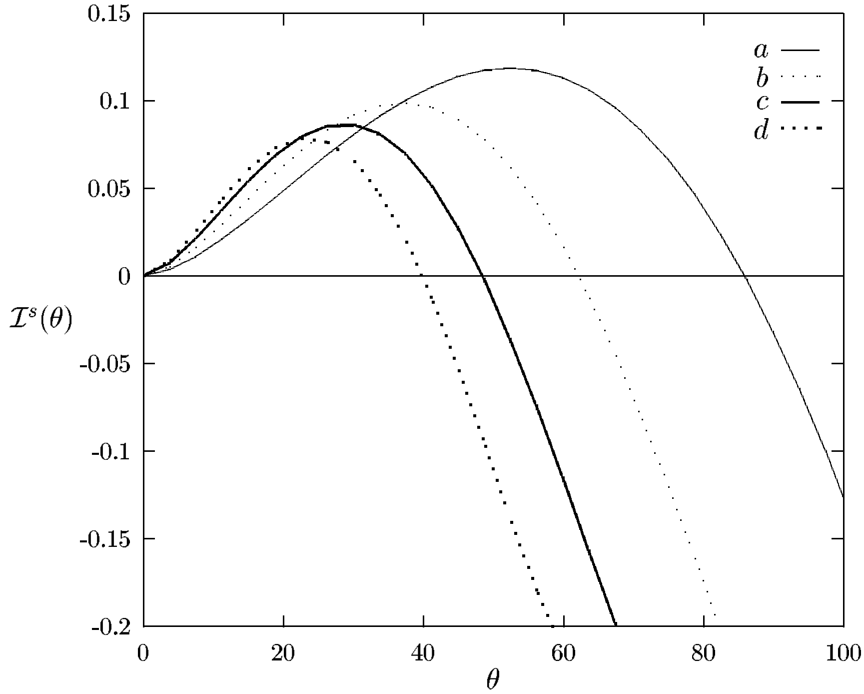

Figure 2. The left-hand side of normalized BC inequality (equation (16)) $\mathcal{I}^s(\theta)$ as a function of the angle θ for different values of spin in the region of violation.The inequality is violated if $\mathcal{I}^s(\theta)$ is positive. Curve a: spin-$\frac{1}{2}$; curve b: spin-1; curve c: spin-$\frac{3}{2}$; curve d: spin-2.

is this *hidden overall factor* $2\log_2(2s+1)$ in the original BC inequality which gives rise to the increasing size of violation as spin value increases.

To further emphasize this observation, we consider the large-s, small-θ form of $\mathcal{I}^s(\theta)$:

$$\mathcal{I}^s(\theta) \sim \frac{(s\theta)^2}{9\log_2(2s+1)}\left(\log_2\frac{1}{3(s\theta)^2} + \frac{8}{3}\log_2 e\right) \tag{17}$$

which is positive for $(s\theta)^2 \leqslant \frac{e^{\frac{8}{3}}}{3}$. The function $\mathcal{I}^s(\theta)$ reaches a maximum when $(s\theta_0)^2 = \frac{e^{\frac{5}{3}}}{3}$ giving rise to the maximum strength of violation as

$$\mathcal{I}^s(\theta_0) = \frac{e^{\frac{5}{3}}\log_2 e}{27\log_2(2s+1)} \tag{18}$$

in the large-s, small-θ limit. It could be observed that, due to the presence of the factor $\log_2(2s+1)$ in the denominator, the size of violation decreases with the increase of spin s.

In general, for four different pairs of analyser orientations $(\vec{a}, \vec{b}')$, $(\vec{a}', \vec{b}')$, $(\vec{a}', \vec{b})$ and $(\vec{a}, \vec{b})$, the normalized BC inequality is given by

$$H_N(\vec{a};\vec{b}') + H_N(\vec{b}';\vec{a}') + H_N(\vec{a}';\vec{b}) - H_N(\vec{a};\vec{b}) \leqslant 1 \tag{19}$$

where $H_N(\vec{a};\vec{b}')$, $H_N(\vec{b}';\vec{a}')$, $H_N(\vec{a}';\vec{b})$, $H_N(\vec{a};\vec{b})$, are the mutual information entropies normalized to their maximum value $2\log_2(2s+1)$. Observe that the four parts of mutual information entropies of the normalized BC inequality involve quantum mechanical joint probabilities, which can be determined from the statistics of experimental runs with four different analyser orientations. We emphasize that the mutual entropies contained in the normalized BC inequality provide an equally suitable physical basis when compared with the conditional entropies involved in the original BC inequality. The significance of the normalized BC inequality is that it exhibits a satisfactory behaviour in the classical limit $s \to \infty$.

The *chained* information inequality of Braunstein and Caves (1988), which extends the inequality equation (13) to involve $N = 2Q$ orientations of the analysers, $\vec{a}_1, \vec{b}_Q, \vec{a}_2, \vec{b}_{Q-1}, \ldots, \vec{a}_{Q-1}, \vec{b}_2, \vec{a}_Q, \vec{b}_1$, assumes, after normalization, the form

$$(N-1)I^s\left(\frac{\theta}{(N-1)}\right) - I^s(\theta) + \left(1 - \frac{N}{2}\right) \leqslant 0 \tag{20}$$

for co-planar geometry (where the vectors $\vec{a}_1, \vec{b}_Q, \vec{a}_2, \vec{b}_{Q-1}, \ldots, \vec{a}_{Q-1}, \vec{b}_2, \vec{a}_{Q_1}$ are co-planar and the successive vectors are separated by an angle $\frac{\theta}{(N-1)}$), and in the limit $N \to \infty$, leads to a simpler form $I^s(\theta) \geqslant \frac{1}{2}$, which is violated for all θ except multiples of π, confirming the earlier observation of Braunstein and Caves (1988).

In conclusion, we have studied information theoretic index of correlation (which is nothing but the normalized mutual information entropy) for EPRB spin-s correlations. The information theoretic BC inequality, when expressed in terms of the correlation index, reveals a overall factor $2\log_2(2s+1)$, which is observed to give rise to the increasing size of violation of the inequality as s increases. We have shown that the size of violation of the *normalized* BC inequality—obtained by dividing the original BC inequality by the factor $2\log_2(2s+1)$—decreases with the increase of spin s, thus restoring a satisfactory behaviour in the classical limit $s \to \infty$.

Acknowledgment

The author would like to thank the referees for their suggestions in the light of which this paper was revised.

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